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THE DYNAMIC BEHAVIOUR OF CYLINDRICAL SHELLS REINFORCED BY RING RIBS—II. SHELLS OF FINITE LENGTH

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Abstract—A circular cylindrical shell reinforced by ribs of limited rigidity and equidistant from each other, clamped by the edges, is under the action of an impulsive load. The basic assumptions are identical to those of part I [1].

When the ribs are placed rather frequently the solution to the problem is obtained from those obtained in part I [1] by a change of notation, so we consider below that the ribs are placed "sparsely". The cases of "small", "medium" and "large" loads are investigated; the differential equations of motion are listed in a form suitable for numerical integration by an electronic computer for every case. As the distance of the spans from the shell's edges varies the dynamical behaviour of the spans is generally different. It is shown by numerical calculation of some concrete examples that *cetera paribus* with sufficient accuracy we may consider the dynamic behaviour of all spans, except extreme ones, identical to that of an infinitely long shell, and the extreme spans moment can be analysed by means of the example of a shell clamped by the edges with one ring in the middle. For the values of the parameter μ satisfying $\mu \geq \mu_1 \leq \sqrt{6}$ this is confirmed analytically.

NOTATION

$2l, R, 2h$	the length of a span, the radius of the middle surface, the thickness of a shell wall, respectively
T_2	the turning force per unit of length carried by a facing
$2\delta, d$	the height and the breadth of a rib, respectively
M_x	the bending moment per unit of the circumference of the middle surface
W	the radial displacement
P	the uniformly distributed pressure
t_0	the duration of pulse action
γ_{01}, γ_{02}	the surface density of the material of a facing and ribs, respectively
x	the coordinate on the generatrix
t	the time
σ_{01}, σ_{02}	the yield limit for stretching–pressing the facing and the ribs, respectively
ν_{01}, ν_{02}	the specific weight of a facing and ribs, respectively
z	the general number of rings of a shell

INTRODUCTION

THE analysis of the dynamic behaviour of an infinitely long reinforced shell carried out in part I [1] was greatly simplified because it was only necessary to consider the behaviour of one span, or, to be more precise, one half of it (due to symmetry).

The situation becomes much more complicated for a shell of a finite length when the shearing forces $q_i^{\pm}$ $(i = 1, 2, \ldots z)$, which, as will be shown below, are far from coinciding with each other, depend on the ordinal number of the rib. Therefore the dynamic behaviour of different links of the shell is different and it becomes necessary to study their interaction during the motion.

In this paper the pecularities are investigated of the dynamic behaviour of cylindrical shells of finite length reinforced by rings. The discussion is based on the hypotheses accepted in part I. For definiteness the boundary conditions are used corresponding to the clamping of the edges.

When ribs are placed frequently enough and their rigidity is comparatively low the solution of the problem can be determined from the solutions obtained in part I by means of the interpretations indicated in Section 6. In this paper the rings are assumed to be placed "sparsely" and to have large but finite rigidity.

1. THE CASE OF "LOW" LOADS

We place the origin of coordinates on the left support (Fig. 1) and we direct the x-axis along the generatrix and the displacements axis W along the internal normal to the middle surface of the shell. Since the case discussed in the sequel pertains to "sparsely" placed rings of comparatively large rigidity, the dynamic motion of the shell has the form shown

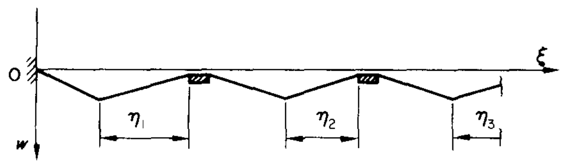

FIG. 1.

in Fig. 1. Let us call a link a part of the shell which encloses a rib and is bounded by the two rings of plastic deformation nearest to the rib. The equation of motion of a link, e.g., for the i-th, can be written in the form (cf. part I [1])

$$m'' + 2\mu^2(t_2 + p - \ddot{w}) = 0 \tag{1.1}$$

$$p + q_i^- + q_i^+ - (1 + a_0) - \lambda \ddot{w}_{\rho i} = 0, \ (i = 1, 2, \ldots, s). \tag{1.2}$$

Here the prime denotes the derivative with respect to the non-dimensional coordinate $\xi = x/l$ and a dot—with respect to the nondimensionless time $\tau = t/t_0$.

Due to the symmetry of the problem only half of the length of the shell is considered; hence

$$s = \begin{cases} z/2, \text{ when } z \text{ is an even integer} \\ (z+1)/2, \text{ when } z \text{ is an odd integer} \end{cases}$$

Just as in part I we consider further the shells with $\mu \geq \mu_*$, where μ_* is determined by the formula (1.5) of part I. Then at the loads $p \leq p_{1*}$ [p_{1*} is defined there by the formula (1.13)] the rings remain motionless and the corresponding motion of the shell spans is investigated in [2]. Therefore we shall further discuss the case of $p \geq p_{1*}$.

For loads slightly exceeding p_{1*} we have the following velocity fields (regime AB in Fig. 2)

$$\dot{w}(\xi,\tau) = \begin{cases} \dot{w}_1\xi/\xi_1 \ldots (0 \le \xi \le \xi_1) \\ \dot{w}_{\rho i} + (\dot{w}_i - \dot{w}_{\rho i})(\xi_i^- - \xi)/(\xi_i^- - \xi_i) \ldots (\xi_i \le \xi \le \xi_i^-) \\ \dot{w}_{\rho i} + (\dot{w}_{i+1} - \dot{w}_{\rho i})(\xi - \xi_i^+)/(\xi_{i+1} - \xi_i^+) \ldots (\xi_i^+ \le \xi \le \xi_{i+1}) \end{cases} \tag{1.3}$$

$$\dot{w}_i = \dot{w}(\xi_i,\tau);\ \dot{w}_{\rho i} = \dot{w}(\xi_i^{\pm},\tau) \qquad (i = 1, 2, \ldots, s).$$

Here ξ_i is the coordinate of the plastic ring hinge of the i-th span, ξ_i^- and ξ_i^+ are the coordinates of the left and the right rims of the same (the i-th) rib. The ordinal number of the ring is considered to coincide with that of the span if the ring is on the right of the span and is one unit greater if the ring is on the left of the span. Substituting (1.3) into (1.1) we obtain after integration of the obtained equation with the boundary conditions

$$m(0,\tau) = m(\xi_i^{\pm},\tau) = -1; \qquad m(\xi_i,\tau) = 1, \tag{1.4}$$

the following field of generalized stresses at $(0 \le \tau \le 1)$

$$m(\xi,\tau) = \begin{cases} \mu^2[\ddot{w}_1\xi^3/3(2-\eta_1) - (p-1)\xi^2] + C_0\xi - 1 \ldots (0 \le \xi \le \xi_1) \\ \mu^2[(\ddot{w}_i - \ddot{w}_{\rho i})(\xi_i^- - \xi)^3/3\eta_i + (\ddot{w}_{\rho i} + 1 - p)\xi^2] + A_i^-\xi + B_i^- \ldots (\xi_i \le \xi \le \xi_i^-) \\ \mu^2[(\ddot{w}_{i+1} - \ddot{w}_{\rho i})(\xi - \xi_i^+)^3/3(2-\eta_{i+1}) + (\ddot{w}_{\rho i} + 1 - p)\xi^2] + A_i^+\xi + B_i^+ \ldots \\ \quad (\xi_i^+ \le \xi \le \xi_{i+1}) \end{cases}$$

$$t_2 = \begin{cases} -1 \ldots (\xi_{i-1}^+ \le \xi \le \xi_i^-) \\ -(1+a_0) \ldots (\xi_i^- \le \xi \le \xi_i^+) \end{cases} \tag{1.5}$$

$$C_0 = -\{\mu^2(2-\eta_1)^2[\ddot{w}_1 - 3(p-1)] - 6\}/3(2-\eta_1);$$

$$A_i^- = \{\mu^2\eta_i[\eta_i(\ddot{w}_i - \ddot{w}_{\rho i}) - 3(\ddot{w}_{\rho i} + 1 - p)(\xi_i + \xi_i^-)] - 6\}/3\eta_i;$$

$$B_i^- = -[A_i^-\xi_i^- + (\mu\xi_i^-)^2(\ddot{w}_{\rho i} + 1 - p) + 1];$$

$$A_i^+ = -\{\mu^2(2-\eta_{i+1})[(\ddot{w}_{i+1} - \ddot{w}_{\rho i})(2-\eta_{i+1}) + 3(\ddot{w}_{\rho i} + 1 - p)(\xi_{i+1} + \xi_i^+)] - 6\}/3(2-\eta_{i+1});$$

$$B_i^+ = -[A_i^+\xi_i^+ + (\mu\xi_i^+)^2(\ddot{w}_{\rho i} + 1 - p) + 1];$$

$$\eta_i = \xi_i^- - \xi_i; \qquad (i = 1, 2, \ldots, s).$$

Here the plastic ring hinges are assumed to be stationary during the action of the load which is described as a rectangular impulse.

In the plastic ring hinges the moment achieves its maximum, therefore

$$m'(\xi_i - 0,\tau) = m'(\xi_i + 0,\tau) = 0. \tag{1.6}$$

Using (1.6), we obtain from (1.5)

$$\mu^2(2-\eta_i)^2[\ddot{w}_{\rho i-1} + 2\ddot{w}_i - 3(p-1)] + 6 = 0$$

$$\mu^2\eta_i^2[\ddot{w}_{\rho i} + 2\ddot{w}_i - 3(p-1)] + 6 = 0, \tag{1.7}$$

whence we obtain the recurrent formula

$$\ddot{w}_{\rho i} = \ddot{w}_{\rho i-1} + 24(\eta_{i-1})/\mu^2\eta_i^2(2-\eta_i)^2; \qquad (i = 1, 2, \ldots, s). \tag{1.8}$$

On the other hand, from equation (1.2) we obtain by applying certain relations (cf. formulae (1.2) in part I)

$$q_i^- = -m'(\xi_i^- - 0, \tau)/2\mu\theta; \qquad q_i^+ = m'(\xi_i^+ + 0, \tau)/2\mu\theta, \quad (i = 1, 2, \ldots, s) \tag{1.9}$$

and from (1.5) it follows that

$$\begin{aligned} a_{\rho i} \equiv \ddot{w}_{\rho i} = & \{4\theta\mu^{-1}(p-1-a_0)+[p-1+6/\mu^2\eta_i(2-\eta_{i+1})](2+\eta_i-\eta_{i+1})\} \\ & \times(4\lambda\theta\mu^{-1}+2+\eta_i-\eta_{i+1})^{-1}. \end{aligned} \tag{1.10}$$

Formulae (1.8) and (1.10) give all in all s algebraic equations for s unknown values of η_i. It is to be considered that here

$$w_{\rho o} \equiv \dot{w}_{\rho 0} \equiv \ddot{w}_{\rho 0} \equiv 0; \; \eta_{s+1} = \begin{cases} 1 \ldots \text{when } z \text{ is an even integer} \\ 2-\eta_s \ldots \text{when } z \text{ is an odd integer} \end{cases} \tag{1.11}$$

Using the zero boundary conditions we obtain from (1.7) and (1.10) the formulae

$$\begin{aligned} w_i(\tau) &= [3(p-1)-a_{\rho i}-6(\mu\eta_i)^{-2}]\tau^2/4 \\ w_{\rho i}(\tau) &= a_{\rho i}\tau^2/2; \qquad (0 \le \tau \le 1). \end{aligned} \tag{1.12}$$

Just as in the case of an infinitely long shell, the bending moment in each span is a cubic function of ξ, hence, to satisfy the conditions $(-1 \le m(\xi, \tau) \le 1)$ it is necessary to satisfy the inequalities

$$m'(0, \tau) \ge 0; \qquad q_i^{\pm}(\tau) \ge 0 \tag{1.13}$$

$$m''(\xi_i \pm 0, \tau) \le 0. \tag{1.14}$$

These conditions will hold if the load p satisfies the inequalities

$$p_{1*} \le p \le p_i^* \tag{1.15}$$

$$p_i^* = 1+6\mu^{-2}\left\{\eta_{i*}^{-2}+\sum_{k=1}^{i}[(2-\eta_{k*})^{-2}-\eta_{k*}^{-2}]\right\} \tag{1.16}$$

where η_{k*} $(k = 1, 2, \ldots, i)$ is calculated from the system (1.8) at $p = p_i^*$ and $\eta_i = \eta_{i*}$ $(i = 1, 2, \ldots, s)$.

As all reinforcing rings are identical and equidistant from one another, it is intuitively clear that at least during the loading we have $\ddot{w}_{\rho i} \ge \ddot{w}_{\rho i-1}$ (this suggestion will be checked later numerically). Then from (1.8) it follows that $\eta_i \ge 1$ $(i = 1, 2, \ldots, s)$, i.e. the plastic hinges of the circumference are shifted from the middle of each span towards the nearest support. Then from (1.16) it follows that $p_{i-1}^* \le p_i^*$, so the violation of the condition $-1 \le m(\xi, \tau) \le 1$ (the appearance of a plastic zone) is first possible in the span adjoining the shell support. The span having one immobile support is more rigid in comparison with the others and as a result the accepted assumption $\ddot{w}_{\rho i} \ge \ddot{w}_{\rho i-1}$ is indirectly confirmed by the results obtained in part I. Indeed, as it is shown there, in case of movable supports the load required for the appearance of a plastic zone is greater than the corresponding load for immobile supports.

Since $p_{i-1}^* \leq p_i^*$, to fulfil the condition $(-1 \leq m \leq 1)$ for all spans of the shell it is sufficient for the load p to satisfy the inequalities

$$p_{1*} \leq p \leq p_1^* = 1+6/\mu^2(2-\eta_{1*})^2 \tag{1.17}$$

When $p > p_1^*$ the inequalities (1.14) are violated in the span which adjoins the support.

After the load has been removed $(\tau \geq 1)$ the shell moves under its own momentum, and the plastic ring hinges shift towards the middle of the span. Functions $\eta_i(\tau)$, $w_i(\tau)$ and $w_{\rho i}(\tau)$ $(i = 1, 2, \ldots, s)$ are determined from the following system of $3s$ second order differential equations

$$\begin{aligned}
\dot{\eta}_i(\tau) &= [(\ddot{w}_{\rho i}+2\ddot{w}_i+3)\eta_i^2+6\mu^{-2}][2\eta_i(\dot{w}_i-\dot{w}_{\rho i})]^{-1};\\
\ddot{w}_{\rho i}(\tau) &= -\{4\theta\mu^{-1}(1+a_0)+[1-6/\mu^2\eta_i(2-\eta_{i+1})](2+\eta_i-\eta_{i+1})\}(4\lambda\theta\mu^{-1}+2+\eta_i-\eta_{i+1})^{-1};\\
\ddot{w}_i(\tau) &= -\{\eta_i(2-\eta_i)[(2-\eta_i)(\dot{w}_i-\dot{w}_{\rho i})(\ddot{w}_{\rho i-1}+3)+\eta_i(\dot{w}_i-\dot{w}_{\rho i-1})(\ddot{w}_{\rho i}+3)]+6\mu^{-2}\\
&\quad\times[\eta_i(\dot{w}_i-\dot{w}_{\rho i})+(2-\eta_i)(\dot{w}_i-\dot{w}_{\rho i-1})]\}\{2\eta_i(2-\eta_i)[\eta_i(\dot{w}_i-\dot{w}_{\rho i-1})\\
&\quad+(2-\eta_i)(\dot{w}_i-\dot{w}_{\rho i})]\}^{-1}\ldots(1 \leq \tau \leq \tau_0).
\end{aligned} \tag{1.18}$$

These equations are obtained by integrating the equation (1.1) with respect to ξ in each link using (1.2), (1.3), (1.4), (1.6) and (1.9). Here $p \equiv 0$.

If z is even the displacement in the middle of the shell is determined by integrating the equation

$$\ddot{w}_{s+1} = -(\ddot{w}_{\rho s}+3p_M)/2; \qquad p_M = 1+2/\mu^2.$$

The system (1.18) with (1.11) taken into account is numerically integrated at any value of s by the Runge–Kutta method. The required $5s$ initial conditions at $\tau = 1$ are calculated from (1.8), (1.10) and (1.12) and have the form

$$\begin{aligned}
&\eta_i(1) = \eta_{0i};\ w_{\rho i}(1) = \dot{w}_{\rho i}(1)/2 = a_{\rho i}/2\\
&w_i(1) = \dot{w}_i(1)/2 = [3(p-1)-a_{\rho i}-6(\mu\eta_{0i})^{-2}]/4 \qquad (i = 1, 2, \ldots, s).
\end{aligned} \tag{1.19}$$

The condition imposed on the velocity,

$$\dot{w}(\xi, \tau) \geq 0, \tag{1.20}$$

determines the upper limit of the integration segment of system (1.18)

$$\tau \leq \tau_0 = \max \tau_{0i}^* \qquad (i = 1, 2, \ldots, s). \tag{1.21}$$

The motion time of the rings adjoining the given span is in the general case different and at the instant that both ends of the span become motionless the plastic ring circumference in this span inevitably reaches its middle. This is the consequence of neglecting the geometrical changes of the shell in its motion because in this case (after both rings stop) the span behaves like a smooth shell with motionless supports. Therefore, if by the time moment τ_{0k}^* k, the rings have stopped their motion, the further integration of system (1.18) leaves $3(s-k)+2$ equations $(k \geq 2)$ if there is no extreme ring among those that have stopped moving, and $3(s-k)$ equations $(k \geq 1)$ if among these there is an extreme one.

Since the rings are identical and equidistant from each other, their different dynamic behaviour is due only to their different distances from a support. Hence, it is assumed that there are no moving rings among these which ended their movement and vice versa of the considered half of a shell.

The generalized stress t_2 is determined by expression (1.5), for $m(\xi, \tau)$ we have

$$m(\xi,\tau) = \begin{cases} \mu^2(\beta_0\xi^3/3+\xi^2)+C_0\xi-1 \ldots (0 \le \xi \le \xi_1) \\ \mu^2[\beta_i^-(\xi_i^- - \xi)^3/3+(\ddot{w}_{\rho i}+1)\xi^2]+A_i^-\xi+B_i^- \ldots (\xi_i \le \xi \le \xi_i^-) \\ \mu^2[\beta_i^+(\xi-\xi_i^+)^3/3+(\ddot{w}_{\rho i}+1)\xi^2]+A_i^+\xi+B_i^+ \ldots (\xi_i^+ \le \xi \le \xi_{i+1}) \end{cases}$$

$$\begin{aligned} C_0 &= -\{\mu^2(2-\eta_1)^2[\beta_0(2-\eta_1)+3]-6\}/3(2-\eta_1); \\ A_i^- &= \{\mu^2[\beta_i^-\eta_i^3-3(\ddot{w}_{\rho i}+1)(2\xi_i+\eta_i)\eta_i]-6\}/3\eta_i; \\ B_i^- &= -[A_i^-\xi_i^- +(\mu\xi_i^-)^2(\ddot{w}_{\rho i}+1)+1]; \\ A_i^+ &= -\{\mu^2(2-\eta_{i+1})[\beta_i^+(2-\eta_{i+1})^2+3(\ddot{w}_{\rho i}+1)(\xi_{i+1}+\xi_i^+)]-6\}/3(2-\eta_{i+1}); \\ B_i^+ &= -[A_i^+\xi_i^+ +(\mu\xi_i^+)^2(\ddot{w}_{\rho i}+1)+1];\ \beta_0 = \frac{d}{d\tau}[\dot{w}_1(2-\eta_1)]; \\ \beta_i^- &= \frac{d}{d\tau}[(\dot{w}_i-\dot{w}_{\rho i})/\eta_i];\ \beta_i^+ = \frac{d}{d\tau}[(\dot{w}_{i+1}-\dot{w}_{\rho i})/(2-\eta_{i+1}). \end{aligned} \tag{1.22}$$

The motion considered in the second phase ($1 \le \tau \le \tau_0$) would hold if inequalities (1.13), (1.14) and $\psi \le 1$ (see inequality 1.9 in part I [1]) were satisfied in all spans of the shell. It can readily be seen that the solution (1.22) holds when only one inequality is true:

$$m'(0, \tau) \ge 0 \tag{1.23}$$

when all rings are moving, and inequality

$$m'(\xi_k^+ + 0, \tau) \ge 0 \tag{1.24}$$

when k rings have stopped moving. It is supposed that rings completing their motions are counted beginning from a support. Inequalities (1.23) and (1.24) are obviously satisfied if

$$\mu_* \le \mu \le \min(\sqrt{6}, \mu_1), \tag{1.25}$$

where μ_1 is defined by the formula (1.14) of part I.

For each span we shall call the third phase of motion the interval $\tau_{0i}^* \le \tau \le \tau_{1i}^0$ from the moment τ_{0i}^* of the full stop of the span ends to the moment τ_{1i}^0 of the full stop of the faces of the span. For each span τ_{0i}^* and τ_{1i}^0 are expected to be different from each other.

The solution of the problem in the third motion phase for each span is obtained just as for the shell with motionless supports [2] but as initial conditions we must use those for displacements and displacement velocities at the time moment $\tau = \tau_{0i}^*$.

The maximal residual displacements in each span is obviously reached in the middle and is determined by the formula

$$w(\xi_i^- - 1, \tau_{1i}^0) = w(\xi_i^- - 1, \tau_{0i}^*) + \dot{w}^2(\xi_i^- - 1, \tau_{0i}^*)/3p_M. \tag{1.26}$$

2. THE CASE OF "OVERAGE" LOADS

Here, as in Section 1, we consider the dynamic behaviour of shells of finite length with the parameter μ in the interval (1.25).

In accordance with what is laid down in Section 1, for loads slightly exceeding the load p_1^* the plasticity conditions (1.14) will first be violated in the first span. Therefore under such loads at the point with the coordinate $\xi_* = 2-\eta_{1*}$ a plastic zone (with regime B, Fig. 2) is formed while in all the other spans the regime AB is maintained. The field of velocities will evidently be determined by formulae (1.3) for $i = 2, 3, \dots, s$, while for the first span we shall have the formulae

$$\dot{w}(\xi,\tau) = \begin{cases} \dot{w}_{01}^{-}\xi/\xi_{01}^{-} \dots (0 \le \xi \le \xi_{01}^{-}) \\ (p-1)\tau \dots (\xi_{01}^{-} \le \xi \le \xi_{01}^{+}) \\ \dot{w}_{\rho 1}+(\dot{w}_{01}^{+}-\dot{w}_{\rho 1})(\xi_1^{-}-\xi)\eta_1^{-1} \dots (\xi_{01}^{+} \le \xi \le 2) \end{cases}$$
$$\dot{w}_{01}^{\pm}(\tau) = \dot{w}(\xi_{01}^{\pm},\tau). \tag{2.1}$$

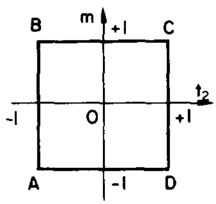

FIG. 2.

Here ξ_{01}^{-} and ξ_{01}^{+} are the boundaries of the plastic zone in the first span. The displacement velocity $\dot{w}(\xi,\tau)$ on the segment $\xi_{01}^{-} \le \xi \le \xi_{01}^{+}$ is obtained by integration of the equation (1.1) with respect to time at $m(\xi,\tau) = 1$ and at the zero initial conditions. In the first span, instead of conditions (1.4), (1.6) and (1.14) we shall have

$$m(\xi_{01}^{-}\pm 0,\tau) = m(\xi_{01}^{+}\pm 0,\tau) = 1 \tag{2.2}$$

$$m'(\xi_{01}^{-}\pm 0,\tau) = m'(\xi_{01}^{+}\pm 0,\tau) = 0 \tag{2.3}$$

$$m''(\xi_{01}^{-}\pm 0,\tau) \le 0; \qquad m''(\xi_{01}^{+}\pm 0,\tau) \le 0. \tag{2.4}$$

When $0 \le \tau \le 1$ the relations (1.5)–(1.8), (1.10)–(1.12) are maintained for all spans, except the first (the expression for t_2 is also valid for the first span). For the first span from (1.1) and (1.2), taking into consideration (1.4), (1.9), (2.1)–(2.3), we shall have

$$m(\xi,\tau) = 1-2(1-y/y_0)^3; \qquad \ddot{w}_{\rho 1} \equiv a_{\rho 1} = p-1-6(\mu\eta_1)^{-2};$$
$$w(\xi,\tau) = \begin{cases} (p-1)y\tau^2/2y_0 \dots (0 \le \xi \le \xi_{01}^{-}) \\ (p-1)\tau^2/2 \dots (\xi_{01}^{-} \le \xi \le \xi_{01}^{+}) \\ (a_{\rho 1}+6y/\mu^2 y_0^3)\tau^2/2 \dots (\xi_{01}^{+} \le \xi \le 2) \end{cases}$$
$$y = \begin{cases} \xi \\ \eta_1, \\ 2-\xi \end{cases} y_0 = \begin{cases} \xi_{01}^{-} \dots (0 \le \xi \le \xi_{01}^{-}) \\ \eta_1 \dots (\xi_{01}^{-} \le \xi \le \xi_{01}^{+}) \\ \eta_1 \dots (\xi_{01}^{+} \le \xi \le 2) \end{cases} \tag{2.5}$$
$$\xi_{01}^{-} = [6/(p-1)]^{\frac{1}{2}}\mu^{-1}; \qquad \eta_1 = 2-\xi_{01}^{+}$$

$$6\lambda(\mu\eta_1)^{-2}-(p-1)(\lambda-1)-a_0+3\{2\eta_1+[\eta_1^2+(2-\eta_2)^2](2-\eta_2)^{-1}\}/2\mu\theta\eta_1^2 = 0. \tag{2.6}$$

Inequalities (1.13) and (2.4) are satisfied, but for (1.14) to be true we require

$$p \leq p_i^{**} = 1+a_{p1}+6\mu^{-2}\left\{\eta_{i*}^{-2}+\sum_{k=2}^{i}\left[(2-\eta_{k*})^{-2}-\eta_{k*}^{-2}\right]\right\}, \tag{2.7}$$

where $\eta_{k*}(k = 2, 3, \ldots, i)$ are calculated from the system (1.8) at $p = p_i^{**}$ and $\eta_i = \eta_{i*}$ with the first equation replaced by (2.6). Since all $\eta_{i*} \geq 1$, the smallest value of p_i^{**} is obviously achieved at $i = 2$ i.e. (2.7) must be replaced by the more strict inequality

$$2-\eta_{2*} \leq \eta_{1*} \tag{2.8}$$

Evidently, inequality (2.8) would be violated when the increase in the load p increases and η_{1*} and η_{2*} become equal to 1. Then, substituting $\eta_{1*} = \eta_{2*} = 1$ into equation (2.6) we shall obtain the upper boundary of the load

$$p_1^* \leq p \leq p_* \tag{2.9}$$

permissible in the case. Here p_* is determined from the formula (2.11) of part I [1].

It can readily be checked, by use of the recurrence formula (1.8) and taking into account (1.10) and (1.11), that the load p determines a dynamic state of the shell in the first phase $(0 \leq \tau \leq 1)$ of its motion when all $\eta_i = 1$.

After the load is removed, the size of the plastic zone formed in the first span is reduced to zero. The corresponding time τ_* is determined from the equation

$$\xi_{01}^-(\tau_*)+\eta_1(\tau_*) = 2. \tag{2.10}$$

In the time interval $1 \leq \tau \leq \tau_*$ the plastic ring circumferences ξ_{01}^+ and $\xi_i(i = 2, 3, \ldots, s)$ move from the middle of the span towards the nearest support. The unknown functions $\eta_i(\tau)$, $w_i(\tau)$ and $w_{pi}(\tau)$ can be found in the interval $1 \leq \tau \leq \tau_*$ by using the system (1.18) again, while the functions $\xi_{01}^-(\tau)$, $\dot{w}_{01}^{\pm}(\tau)$, $\dot{\eta}_1(\tau)$ and $\dot{w}_{p1}(\tau)$ are determined in a similar way, just as in the first phase of the motion considered at this point, and have the form

$$\xi_{01}^-(\tau) = [6\tau(p-\tau)^{-1}]^{\frac{1}{2}}\mu^{-1}; \qquad \dot{w}_{01}^{\pm}(\tau) = p-\tau; \ldots (1 \leq \tau \leq \tau_*) \tag{2.11}$$

$$\dot{\eta}_1 = \eta_1\{3[a+\eta_1+\eta_1^2(2-\eta^2)^{-1}]+2\mu\theta\eta_1^2(\lambda-1-a_0)\}/6\tau a(1 \leq \tau \leq \tau_*);$$

$$a = 4\lambda\theta\mu^{-1}+2+\eta_1-\eta_2 \tag{2.12}$$

$$\dot{w}_{p1}(\tau) = p-\tau-6\tau(\mu\eta_1)^{-2}\ldots(1 \leq \tau \leq \tau_*). \tag{2.13}$$

The stress t_2 has the form (1.5) and $m(\xi, \tau)$ is determined by the same cubic polynomial from ξ as is (1.22), but with other coefficients which also depend on time.

The necessary plasticity conditions at loads (2.9) would be satisfied if μ satisfies the inequality (1.25).

From equation (2.10), taking into account (2.11) and (1.17), we obtain

$$\tau_* = p/p_1^*.$$

It is assumed here that $\eta_1(\tau_*) = \eta_{1*}$ where η_{1*} determined the position of the plastic ring circumference in the first span at $p = p_1^*$. In each concrete problem this assumption can be checked numerically.

The further motion of the shell at $\tau_* \leq \tau \leq \tau_{1i}^0$ is described by the scheme of Section 1 over the time interval $1 \leq \tau \leq \tau_{1i}^0$. In this case it is necessary to use the initial conditions at $\tau = \tau_*$.

3. THE CASE OF "HIGH" LOADS

Let us consider the dynamic behaviour of the shell under the loads (3.1) of part I. In this case the dynamic behaviour of all the links of the shell whose parameter satisfies inequality (1.25) to the time moment $\tau = \tau_1 = p/p_*$ cf. formula (3.10) of part I would be the same as in an infinitely long shell, and the motion of the segment $(0 \leq \xi \leq \xi_{01}^-)$ is determined by the solutions (24), (25) [2].

At $\tau \geq \tau_1$ the motion proceeds in a way similar to that described in Section 2 only instead of the initial conditions at $\tau = 1$, the corresponding initial conditions at $\tau = \tau_1$ must be used.

It should be noted that by the time moment $\tau = \tau_1$ the plastic zone reduces to a point just in the middle of each span except the first one. In the first span the zone still exists, the coordinate of its boundary near the support being determined at the time moment $\tau = \tau_1$ by the expression

$$\xi_{01}^-(\tau_1) = [6(p_* - 1)^{-1}]^{\frac{1}{2}}\mu^{-1}$$

and the far boundary being also located in the middle of the span. As the motion continues the plastic zone reduces to zero.

But if the parameter μ satisfies the inequality

$$\mu \geq \mu_1 \leq \sqrt{6} \tag{3.1}$$

the behaviour of all shell links over the entire period of motion would be the same as that of an infinitely long shell, while the behaviour of the section $(0 \leq \xi \leq 1)$ coincides with that described in [2]. In this case the maximum residual displacements in all spans are equal and are determined by the formulaes (3.14) and (5.5) of part I [1].

If the parameter μ satisfies the inequality

$$\mu \geq \mu_1 \geq \sqrt{6}$$

then, as in [2], after the load is removed a plastic zone is formed in the shell near the support, which increases starting from zero and where there is no motion (regime *AD*). After $\tau = \tau_0$ [cf. the formula (3.10) of part I] a similar situation is observed near each ring. Just as at μ from (3.1), in this case all rings stop at the same time $\tau = \tau_0$, and the plastic zone boundaries in each span, except the first one are at this instant equidistant from the middle of the span. As the motion proceeds with the motionless rings, it will coincide with [2] in all spans but the first one. The dynamic behaviour of the first span can be analyzed on an example of a smooth edge-clamped shell with one rib in the middle.

Similarly, when

$$\max(\sqrt{6}, \mu_*) \leq \mu \leq \mu_1$$

after the load is taken off near the support there appears a zone in the shell where is no motion. But in this case the motion of the remaining part of the shell proceeds in accordance with the amount of the load both in the presence of the plastic zone and that of plastic hinges.

The results of the calculation performed by the electronic computer M-20 are listed below in Tables 1–6. The calculation was performed with the following geometric and plastic parameters of the shell fixed $\mu = 2$, $a_0 = 9$, $\theta = 0{\cdot}25$; the face and ribs are supposed to be of one material $(\lambda = 10)$. When the parameters μ, a_0 and θ take other values, the relations we get are identical to that of Tables 1–6 and hence are not given here.

TABLE 1

	Load p	$z = 5$			$z = 3$		$z = 1$
		η_1	η_2	η_3	η_1	η_2	η_1
	$p_{1*} = 1{\cdot}6000$	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	1·8200	1·012869	1·000135	1·000004	1·012869	1·000115	1·012868
	2·0500	1·026396	1·000348	1·000005	1·026396	1·000343	1·026392
	2·2700	1·039378	1·000376	1·000006	1·039378	1·000372	1·039250
	2·49	1·052377	1·000399	1·000008	1·052368	1·000402	1·052358
$z = 5$	$p_1^* =$ 2·719122	1·065902	1·000345	1·000012	—	—	—
$z = 3$	$p_1^* =$ 2·719106	—	—	—	1·065901	1·000516	—
$z = 1$	$p_1^* =$ 2·718503	—	—	—	—	—	1·065703
	2·7700	1·053047	1·000090	1·000009	1·053042	1·000089	1·053040
	2·8600	1·031349	1·000033	1·000000	1·031229	1·000028	1·031200
	$p_* = 3{\cdot}0000$	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	3·5000	0·906347	0·906347	0·906347	0·906347	0·906347	0·906347
	4·0000	0·833333	0·833333	0·833333	0·833333	0·833333	0·833333
	5·0000	0·725880	0·725880	0·725880	0·725880	0·725880	0·725880

Table 1 illustrates the behaviour of $\eta_i = \eta_i(p)$ in the first phase of the motion for values of z equal to 5, 3 and 1. The functions $\eta_i(p)$ are increasing with increasing p, in the interval $p_{1*} \leq p \leq p_1^*$, up to maximum values at $p = p_1^*$. When $p > p_1^*$ all functions $\eta_i(p)$ are decreasing; when $p = p_*$ all η_i are equal to unity. In this manner the earlier accepted hypothesis is confirmed that in the first phase of the motion we have $\ddot{w}_i \geq \ddot{w}_{i-1}$. It is worth noting also that $\eta_i(p)$ in practice does not depend on z and that for each z all η_i except η_1 may be put equal to unity with a sufficient degree of accuracy.

In the Tables 2 and 3 the functions $\eta_i(\tau)$ are listed for $z = 3$ and $z = 1$ for "low" $p = 2{\cdot}49$ and "high" $p = 4{\cdot}00$ loads. All functions $\eta_i(\tau)$ are decreasing functions of time when a given load is "low"; when the load is "high"—all are increasing up to the time $\tau = \tau_*$, and for $\tau > \tau_*$, $\eta_i(\tau)$ are decreasing. This behaviour numerically confirms the formula $\tau_* = p/p_1^*$,

TABLE 2

Time τ	$p = 2{\cdot}49$		
	$z = 3$		$z = 1$
	η_1	η_2	η_1
1·00	1·052368	1·000402	1·052358
1·05	1·043356	1·000391	1·043117
1·10	1·034243	1·000380	1·034055
1·15	1·025001	1·000353	1·024887
1·20	1·015644	1·000301	1·015602
1·25	1·006152	1·000281	1·006003
1·30	0·996506	1·000253	0·996412
1·35	0·986686	1·000210	0·986417
1·40	0·976666	1·000146	0·976555
1·45	0·966409	1·000111	0·966308
1·50	0·955863	1·000058	0·955719
$\tau_{01}^* = \tau_{02}^* = 1{\cdot}55$	0·944946	1·000009	0·944981
$\tau_{01}^* = 1{\cdot}56$	—	—	0·944876

TABLE 3

Time τ	$p = 4.00$		
	$z = 3$		$z = 1$
	η_1	η_2	η_1
1·00	0.833333	0·833333	0·833333
1·10	0·986173	0·986173	0·986173
1·20	1·003997	1·000175	1·003815
1·30	1·021188	1·000296	1·021063
$\tau_{*3} = 1{\cdot}470$	1·065901	1·000516	1·065684
$\tau_{*1} = 1{\cdot}472$	1·064873	1·000494	1·065702
1·60	1·047618	1·000452	1·046916
1·70	1·032317	1·000403	1·032001
1·80	1·021109	1·000351	1·020916
1·90	1·009211	1·000302	1·009010
2·00	1·000334	1·000259	1·000298
2·10	0·992406	1·000208	0·991309
2·20	0·971785	1·000161	0·970986
2·30	0·952305	1·000109	0·951978
$\tau_{01}^* = \tau_{02}^* = 2{\cdot}47$	0·942832	1·000054	0·942816
$\tau_{02}^* = 2{\cdot}49$	—	—	0·941899

which is reflecting the fact of vanishing of the plastic zone in the first span, when $\tau = \tau_*$. Note that at the time $\tau = \tau_*$ the plastic zone reduces to a point just in that section of the span where it first may appear when a lower load is applied. The time of the motion is the same for all rings in the case (with the exactness of the integration's step, taken up equal in the problem $\Delta\tau = 0{\cdot}01$). We must note that by Tables 2 and 3 we have till the end

TABLE 4

Time τ	$p = 2{\cdot}49$			
	$z = 3$		$z = 1$	Infinitely long shell
	w_1	w_2	w_1	w
1·00	0·6793	0·6698	0·6799	0·6630
1·05	0·7680	0·7278	0·7692	0·7268
1·10	0·8310	0·7875	0·8319	0·7858
1·15	0·8887	0·8422	0·8897	0·8398
1·20	0·9413	0·8921	0·9423	0·8882
1·25	0·9886	0·9370	0·9902	0·9331
1·30	1·0375	0·9770	1·0397	0·9724
1·35	1·0668	1·0121	1·0689	1·0068
1·40	1·0992	1·0373	1·1008	1·0362
1·45	1·1256	1·0636	1·1273	1·0608
1·50	1·1467	1·0823	1·1484	1·0804
1·55	1·1626	1·0970	1·1639	1·0951
1·56	1·1638	1·0984	1·1643	1·0959
1·60	1·1788	1·1043	1·1804	1·1062
$\tau_{11}^0 = 1{\cdot}66;\ \tau_{12}^0 = 1{\cdot}65$	1·1816	1·1208	—	—
$\tau_{11}^0 = 1{\cdot}66$	—	—	1·1845	1·1088

TABLE 5

Time τ	$p = 4{\cdot}00$			
	$z = 3$		$z = 1$	Infinitely long shell
	w_1	w_2	w_1	w
1·00	1·6217	1·5016	1·6229	1·5000
1·10	1·9611	1·7992	1·9627	1·7950
1·20	2·2443	2·0881	2·2461	2·0800
1·30	2·5274	2·3602	2·5288	2·3550
$\tau_{*3} = 1{\cdot}470$	3·1781	2·8043	2·9597	2·7905
$\tau_{*1} = 1{\cdot}472$	3·1794	2·8056	2·9640	2·7953
1·60	3·3045	3·0889	3·2203	3·0857
1·70	3·5420	3·2943	3·4267	3·2900
1·80	3·7518	3·4794	3·5617	3·4749
1·90	3·9365	3·6449	3·7376	3·6401
2·00	4·0743	3·7897	3·8224	3·7855
2·10	4·1927	3·9151	3·9560	3·9114
2·20	4·3429	4·0283	3·9927	4·0175
2·30	4·4247	4·1177	4·0424	4·1041
$\tau^*_{01} = \tau^*_{02} = 2{\cdot}47$	4·5168	4·2178	4·0914	4·2062
$\tau^*_{01} = 2{\cdot}49$	4·5186	4·2193	4·1047	4·2145
2·60	4·5379	4·2375	4·1804	4·2283
$\tau^0_{11} = 2{\cdot}69$; $\tau^0_{12} = 2{\cdot}67$	4·4812	4·2542	—	—
$\tau^0_{11} = 2{\cdot}69$	—	—	4·4857	—

of the motion $\eta_2 \approx 1$ while $\eta_1 \neq 1$. As the change of geometry was neglected the movement is further impossible at $\eta_1 \neq 1$; so η_1 was put equal to unity while we considered the further movement; i.e. it was assumed that the first type jump of the function $\eta_i(\tau)$ holds at the instant that the ring stops by the 6–7 per cent of magnitude.

It is natural to consider, *cetera paribus*, that the different dynamic behaviour of links with rings of limited rigidity of the infinitely long shell and the shell of finite length is

TABLE 6

Load p		$z = 3$		$z = 1$	Infinitely long shell
		w_1	w_2	w_1	w
	$p_{1*} = 1{\cdot}6000$	0·0800	0·0800	0·0800	0·0800
	1·8200	0·2874	0·2703	0·2888	0·2688
	2·0500	0·5481	0·5186	0·5509	0·5122
	2·2700	0·8465	0·7911	0·8486	0·7892
	2·4900	1·1816	1·1208	1·1845	1·1088
$z = 3$	$p^*_1 = 2{\cdot}7191$	1·6021	1·4901	—	1·4878
$z = 1$	$p^*_1 = 2{\cdot}7185$	—	—	1·5889	1·4865
	2·8600	1·8785	1·7498	1·8812	1·7437
	3·0000	2·1716	2·0212	2·1739	2·0157
	3·5000	3·2503	3·0397	3·2529	3·0336
	4·0000	4·4812	4·2542	4·4857	4·2480
	5·0000	7·8314	7·2719	7·8355	7·2625
	7·0000	16·5748	15·7841	16·5886	15·6345

due to the magnitude of the shift of η_i from unity. Hence we must expect (by Tables 1–3) the maximum residual displacements of each span, in the limits of the assumptions to be slightly different from those of an infinitely long shell. This is true for all spans except extremes and for these is the maximum residual displacements coinciding with those of a shell with one ring in the middle ($z = 1$). Numerical results listed in the Tables 4–6 confirm the statement.

In the Tables 4 and 5 maximum displacements are listed as functions of time, correspondingly to "low" $p = 2{\cdot}49$ and "high" $p = 4{\cdot}00$ loads. The time of stop of the skin-plating τ_{11}^0 and τ_{12}^0 are not equal. The difference is implied by the jump of $\eta_1(\tau)$ at the stop-time of the ring. In Table 6 maximum residual displacements are listed in dependence of the load p.

It is seen from Tables 4–6 that maximum displacements of non extreme spans are exceeding by 6–8 per cent those of intrinsic spans.

Remember that in part I of this work [1] it was shown that maximum residual displacements of a shell with identical movable supports are less (though very slightly) than those of a shell with a fixed support. An analogous situation holds in this case. Extreme spans correspond to the case of shells having one movable and one fixed edge. The percentage difference is greater in this case than that for shells with symmetrically fixed edges.

So numerical calculations show that maximum residual displacements in all spans of a shell of finite length, except extremes in practice coincide with those of an infinitely long shell; in the extremes with a maximum residual displacement of a shell with one ring in the middle.

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Абстракт—Круговая цилиндрическая оболочка, подкрепленная равноотстоящими друг от друга кольцевыми ребрами ограниченной жесткости и защемленная по краям, подвержена действию импульсивной нагрузки. Исходные предположения идентичны принятым в части I [1].

Так как при достаточно большой частоте расположения ребер решение задачи получается из решений, найденных в части I [1], путем указанных там в пункте 6 переобозначений, то ниже считается, что кольца расположены "редко". Исследуются случаи "низких", "средних" и "высоких" нагрузок, для каждого случая выписаны дифференциальные уравнения движения в виде, удобном для интегрирования на ЭЦВМ.

Вследствие неодинаковости расположения пролетов от краев оболочки динамическое поведение их, вообще говоря, является различным. Однако, на конкретных примерах численным путем показывается, что при прочих равных условиях с достаточной степенью точности можно считать поведение всех пролетов, кроме крайних, таким же как и в бесконечно длинной оболочке, а движение крайних пролетов можно проанализировать на примере защемленной по краям оболочки с одним кольцом в середине. Для параметра μ в интервале $\mu \geqslant \mu_1 \leqslant \sqrt{6}$ этот факт подтверждается аналитически.